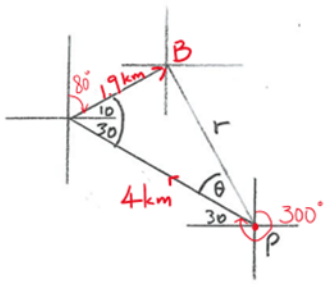
Question 1 **(9 marks)**

(a) A sailing boat leaves port and sails on a bearing of 300° for 4 km before turning to sail 1.9 km on a bearing of 080°. How far is the boat from the port and what is the bearing of final position of the boat from the port. (3 marks)



$r = \sqrt{4^2 + 1.9^2 - 2 \times 4 \times 1.9 \cos 40°}$

$= 2.82 \text{ km}$

$\frac{\sin\theta}{1.9} = \frac{\sin 40°}{2.82}$

$\theta = \sin^{-1}\left(\frac{1.9 \times \sin 40°}{2.82}\right) = 25.64°$

Bearing of the boat (B) from the port (P) is $30° + 25.64°$

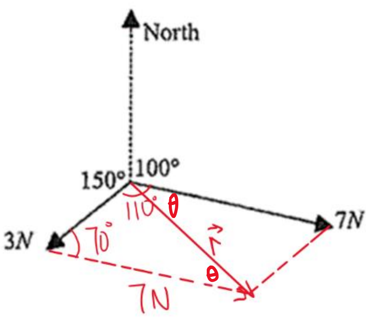
$\approx 326° T$

✓ Uses cosine rule

✓ Uses sine rule

✓ Correct bearing

(b) Find the magnitude and direction of the resultant of the pair of forces in the diagram below. (3 marks)



$|\vec{r}| = \sqrt{7^2 + 3^2 - 2 \times 7 \times 3 \cos 70°}$

$= 6.6 N$

$\frac{\sin\theta}{3} = \frac{\sin 70°}{6.6}$

$\theta = \sin^{-1}\left(\frac{3 \times \sin 70°}{6.6}\right) = 25.3°$

$\therefore$ Bearing $= 100° + 25.3° \approx 125° T$

✓ Correct magnitude of resultant vector

✓ Uses sine rule to find θ

✓ Correct bearing

(c) In still air, an aircraft can maintain a speed of 350 km/h. In what direction should the aircraft be pointing if it wished to travel in a direction 160°, and a 35 km/h wind is blowing from 075°? (3 marks)

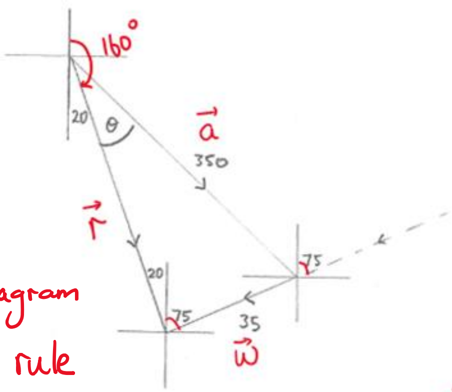
$\vec{a}$ is the velocity of the aircraft.

$\vec{w}$ is the velocity of the wind.

$\vec{r}$ is the resultant velocity.

$\frac{\sin\theta}{35} = \frac{\sin 95°}{350}$

$\theta = \sin^{-1}\left(\frac{35 \times \sin 95°}{350}\right) = 5.7°$



✓ Correct diagram

✓ Uses sine rule to find θ

✓ Correct bearing

$\therefore$ Bearing of $\vec{a} = 160° - 5.7° \approx 154° T$

Question 2 **(6 marks)**

(a) Vectors **a** and **b** have the same magnitude and vectors a and c are perpendicular, where a = $\begin{bmatrix} m \\ n \end{bmatrix}$, b = $\begin{bmatrix} -4 \\ 6 \end{bmatrix}$ and c = $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Determine the values of m and n. (3 marks)

$$\begin{cases} m^2 + n^2 = (-4)^2 + 6^2 = 52 \\ 2m + 3n = 0 \end{cases}$$

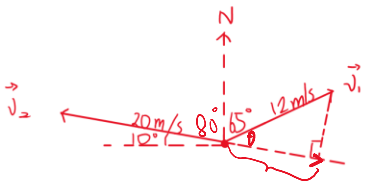
$$\therefore \begin{cases} m = -6 \\ n = 4 \end{cases} \qquad \begin{cases} m = 6 \\ n = -4 \end{cases}$$

✓ Correct equations of m and n

✓ Correct values of m

✓ Correct values of n

(b) Determine the scalar projection of a velocity of 12 m/s on a bearing of 065° onto a velocity of 20 m/s on a bearing of 280°, giving your answer to two decimal places. (3 marks)



$-12 \times \cos(180° - 80° - 65°)$ or $12 \times \cos(80° + 65°)$

$= -12 \times \cos 35°$ or $12 \times \cos 145°$

$= -9.83$

✓ correct angle

✓ correct direction (negative)

✓ correct scalar projection

Question 3 **(8 marks)**

(a) A triangle has vertices as A$(-3,1)$, B$(-1,4)$ and C$(5,0)$.

(i) Determine the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$. (2 marks)

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\overrightarrow{AC} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

-1 each error

(ii) Use a vector method to prove that triangle ABC is right-angle. (2 marks)

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$= 12 - 12$$

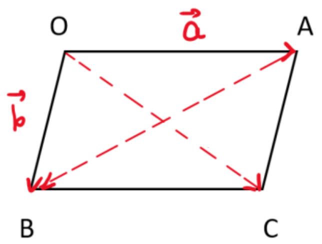
$$= 0$$

$\therefore \overrightarrow{AB} \perp \overrightarrow{BC} \Rightarrow \triangle ABC$ right-angled at B.

✓ Expresses $\overrightarrow{AB} \cdot \overrightarrow{BC}$

✓ Evaluates $\overrightarrow{AB} \cdot \overrightarrow{BC}$

(b) Use a vector method to prove that if the diagonals of a parallelogram are perpendicular to each other, then the parallelogram is a rhombus. (4 marks)



Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$,

so $\overrightarrow{OC} = \vec{a} + \vec{b}$, $\overrightarrow{AB} = \vec{b} - \vec{a}$

If $\overrightarrow{OC}$ & $\overrightarrow{AB}$ are perpendicular, then

$$\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$$

$$(\vec{a} + \vec{b})(\vec{b} - \vec{a}) = 0$$

$$\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - \vec{a}\cdot\vec{b} = 0$$

$$\vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b}$$

$$\therefore |\vec{a}|^2 = |\vec{b}|^2$$

$$\therefore |\vec{a}| = |\vec{b}|$$

$$\therefore |\overrightarrow{OA}| = |\overrightarrow{OB}|$$

Hence, ▱OACB is a rhombus

✓ States $\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$

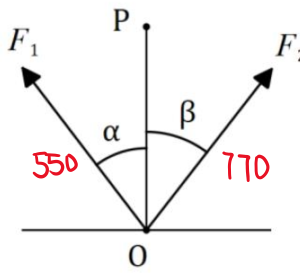
✓ States $\overrightarrow{OA} \cdot \overrightarrow{OA} = \overrightarrow{OB} \cdot \overrightarrow{OB}$

✓ States $|\overrightarrow{OA}|^2 = |\overrightarrow{OB}|^2$

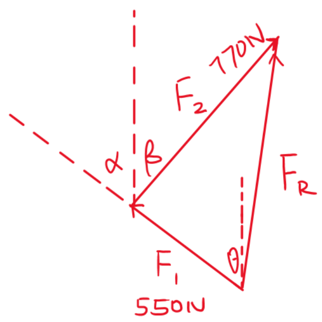
✓ States $|\overrightarrow{OA}| = |\overrightarrow{OB}|$

Question 4 **(8 marks)**

Two forces, F_1 = 550N and F_2 = 770N, act on a body at O, and make angles of $\alpha = 33°$, and $\beta = 18°$ respectively with the vertical OP, as shown in the diagram below.



(a) Determine the magnitude of resultant force and the angle it makes with the vertical. (5 marks)



$$F_R = \sqrt{550^2 + 770^2 - 2 \times 550 \times 770 \times \cos(180-33-18)}$$

$$= 1195.17N$$

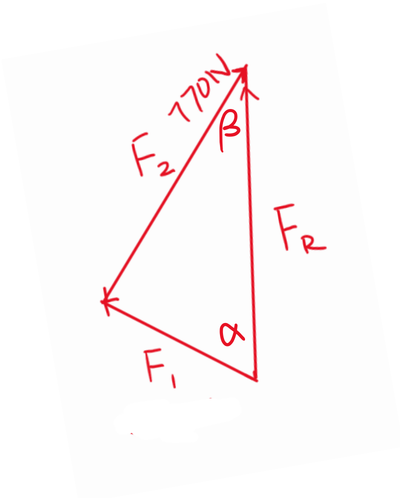
$$\frac{\sin\theta}{770} = \frac{\sin 129°}{1195.17}$$

$$\theta = 30°$$

Required Angle $= 33° - 30°$
$= 3°$

√ Sketch with force nose to tail
√ Indicate use of cosine rule for magnitude
√ Correct magnitude
√ Indicate use of sine rule for angle
√ Angle with vertical

(b) The magnitude of F_1 is to be adjusted so that the direction of the resultant is vertical. Determine the required magnitude of F_1. (3 marks)



$$\frac{\sin\alpha}{F_2} = \frac{\sin\beta}{F_1}$$

$$F_1 = \frac{770 \times \sin 18°}{\sin 33°} = 436.88\text{ N (2 d.p.)}$$

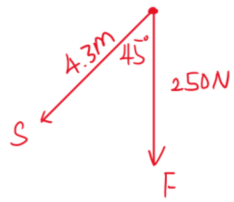
≈ 437N

√ Sketch
√ Indicate use of sine rule
√ Correct magnitude

Question 5 **(9 marks)**

(a) The work done, in joules, by a force of **F** Newtons in changing the displacement of an object by **s** metres, is given by the scalar product of **F** and **s**.

(i) A force of 250 N acting due south moves an object 4.3 m in a south-westerly direction. Determine the work done. (2 marks)



$$W = \vec{s} \cdot \vec{F}$$
$$= |\vec{s}||\vec{F}|\cos 45°$$
$$= 4.3 \times 250 \times \cos 45°$$
$$= 760 \text{ J}$$

✓ Substitute correctly

✓ Evaluate correctly

(ii) Another force of 155 N does 269 joules of work in moving an object 190 cm. Determine the angle between the force and the direction of movement. (2 marks)

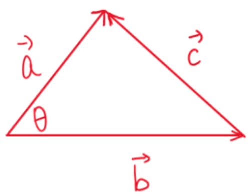
$$155 \times 1.9 \times \cos\theta = 269$$
$$\theta = \cos^{-1}\left(\frac{269}{155 \times 1.9}\right)$$
$$= 24°$$

✓ Substitude correctly

✓ Evaluate correctly

(b) A triangle is formed by three non-zero vectors **a**, **b** and **c**, so that **c** = **a** − **b**, and θ is the angle between **a** and **b**.

(i) Sketch the triangle. (1 mark)



✓ Sketch

(ii) Explain why $\mathbf{c} \cdot \mathbf{c} = |\mathbf{c}|^2$. (1 mark)

$$\vec{c} \cdot \vec{c} = |\vec{c}||\vec{c}|\cos\theta$$

Since $\theta = 0°$, $\cos\theta = 1$

Hence, $\vec{c} \cdot \vec{c} = |\vec{c}|^2$

✓ Explanation

(iii) Use $\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$ to deduce the cosine rule. (3 marks)

✓ Expand scalar product

✓ Use $\vec{c} \cdot \vec{c} = |\vec{c}|$ from (ii)

✓ Use scalar product definition

$$\vec{c} \cdot \vec{c} = (\vec{a} - \vec{b})(\vec{a} - \vec{b})$$
$$\vec{c} \cdot \vec{c} = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$$
$$\therefore |\vec{c}|^2 = |\vec{a}|^2 - |\vec{a}||\vec{b}|\cos\theta - |\vec{b}||\vec{a}|\cos\theta + |\vec{b}|^2$$

Hence, $|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}| - 2|\vec{a}||\vec{b}|\cos\theta$

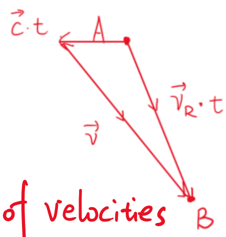
Question 6 **(11 marks)**

A small boat that can maintain a steady speed of 5 ms^{-1} is to cross a river from A to B, where $\overrightarrow{AB} = (35\mathbf{i} - 105\mathbf{j})$ m.

A current of $(-\mathbf{i} - 2\mathbf{j})$ ms^{-1} flows in the river.

The velocity vector that the pilot of the small boat must set to travel from A to B is $a\mathbf{i} + b\mathbf{j}$, where a and b are constants.

(a) Explain why $t(a-1) = 35$ and $t(b-2) = -105$, where t is a constant. (3 marks)



Resultant velocity $\vec{v}_R = \vec{v} + \vec{c} = (a\vec{i} + b\vec{j}) + (-\vec{i} - 2\vec{j})$

$= (a-1)\vec{i} + (b-2)\vec{j}$ is parallel to $\overrightarrow{AB}$

$\therefore \overrightarrow{AB} = \vec{v}_R \cdot t = \left((a-1)\vec{i} + (b-2)\vec{j}\right)t$

$= t(a-1)\vec{i} + t(b-2)\vec{j} = 35\vec{i} - 105\vec{j}$

Hence, $t(a-1) = 35$ & $t(b-2) = -105$

✓ Use sum of velocities

✓ Use equation for parallel condition

✓ Equate individual coefficients

(b) Eliminate t from the equations in (a) and hence express b in terms of a, simplifying your expression. (3 marks)

$\because t(a-1) = 35 \Rightarrow t = \dfrac{35}{a-1}$ & $t(b-2) = -105 \Rightarrow t = \dfrac{-105}{b-2}$

$\therefore \dfrac{35}{a-1} = \dfrac{-105}{b-2}$

$35b - 70 = -105a + 105$

$b - 2 = -3a + 3$

$b = -3a + 5$

✓ Eliminate t

✓ Cross multiply

✓ Simplify

(c) Explain why $a^2 + b^2 = 25$. (1 mark)

The magnitude of $\vec{v} = a\vec{i} + b\vec{j}$ is $\sqrt{a^2 + b^2} = \sqrt{25}$

Hence the speed of the small boat is 5 m/s

✓ State magnitude and speed

(d) Use your equations from (b) and (c) to determine the values of a and b. (3 marks)

$$\begin{cases} b = -3a + 5 \\ a^2 + b^2 = 25 \end{cases} \Rightarrow a^2 + (-3a+5)^2 = 25$$

$$\Rightarrow a = 3 \text{ or } a = 0 \text{ (exclude)}$$

$$\therefore b = -3 \times 3 + 5 = -4$$

$$\therefore \vec{v} = 3\vec{i} - 4\vec{j}$$

✓ Correct equation

✓ Solve a & b

✓ Eliminate alternative solution

(e) Determine the time that the small boat will take to travel from A to B. (1 mark)

$$t = \frac{35}{a-1} = \frac{35}{2} = 17.5\text{s}$$

✓ Correct time

Question 7 **(8 marks)**

Three vectors are given by $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = -3\mathbf{i} + 1.5\mathbf{j}$ and $\mathbf{c} = -2\mathbf{i} + y\mathbf{j}$, where y is a constant.

(a) Determine the vector projection of $\mathbf{b}$ on $\mathbf{a}$. (3 marks)

$$\vec{b} \cdot \vec{a} = 3 \times (-3) + (-4) \times (1.5) = -15$$

$$\vec{a} \cdot \vec{a} = 3^2 + (-4)^2 = 25$$

$$\vec{u} = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{-15}{25} \times (3\vec{i} - 4\vec{j})$$

$$= -\frac{9}{5}\vec{i} + \frac{12}{5}\vec{j}$$

✓ Evaluate $\vec{b} \cdot \vec{a}$ & $\vec{a} \cdot \vec{a}$

✓ State $\vec{u} = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$

✓ Correct vector projection

(b) Determine the value(s) of y if

(i) $\mathbf{a}$ and $\mathbf{c}$ are perpendicular. (2 marks)

$$\vec{a} \cdot \vec{c} = 0$$

$$3 \times (-2) - 4y = 0$$

$$y = -\frac{3}{2}$$

✓ Use scalar product

✓ Solve y

(ii) the angle between the directions of $\mathbf{b}$ and $\mathbf{c}$ is 45°. (3 marks)

$$\vec{b} \cdot \vec{c} = 6 + 1.5y$$

$$|\vec{b}| = \sqrt{(-3)^2 + 1.5^2} = \frac{3\sqrt{5}}{2}$$

$$|\vec{c}| = \sqrt{(-2)^2 + y^2} = \sqrt{4 + y^2}$$

$$\therefore \vec{b} \cdot \vec{c} = |\vec{b}||\vec{c}| \cos 45^\circ$$

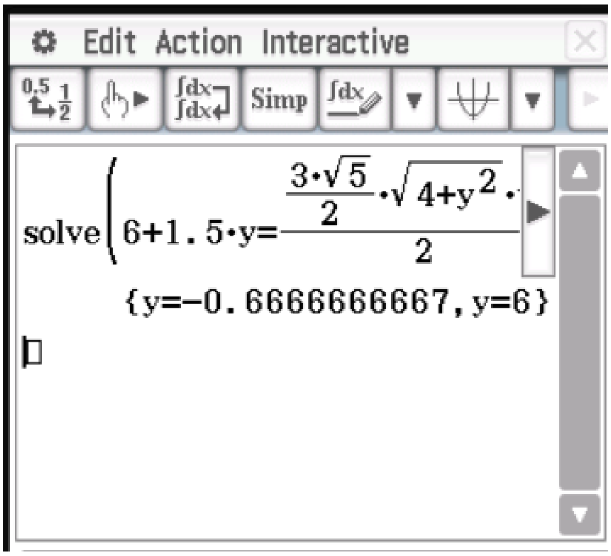
$$6 + 1.5y = \frac{3\sqrt{5}}{2} \times \sqrt{4 + y^2} \times \frac{\sqrt{2}}{2}$$

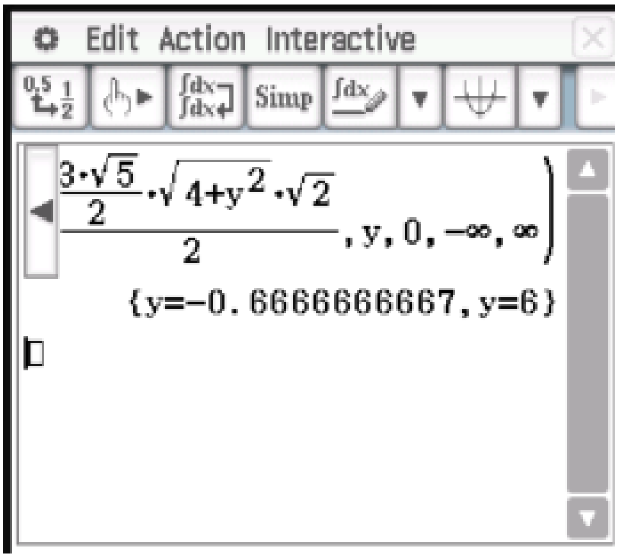
$$y = -\frac{2}{3}, \quad y = 6$$

✓ Use scalar product

✓ State one solution

✓ State second solution






```
[−3, 1.5]⇒b
                                  [−3  1.5]
[−2, y]⇒c
                                    [−2  y]
norm(b)×norm(c)×cos(45)=dotP(b,c)
```

$$0.75\cdot\left(10\cdot\left(y^2+4\right)\right)^{0.5}=1.5\cdot y+6$$

```
solve(ans, y, 0, −∞, ∞)
                 {y=−0.6666666667, y=6}
```